A LARGE SIEVE FOR A CLASS OF NON-ABELIAN L-FUNCTIONS

BY

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ABSTRACT

Let q be a fixed odd prime. We consider the sequence of Kummer fields $Q(\sqrt[q]{1}, \sqrt[q]{a})$ as a varies. Estimates are given for the global density of zeroes of Artin L-functions of these fields. These results are obtained by deducing a series representation for the Artin L-functions that arises naturally in the arithmetic of Q.

1. Introduction

Owing to the fact that the zeta-functions of abelian extensions of the rational number field factor into a product of L-functions, it is possible to deduce results about their distribution of zeros that would not otherwise be obtained by a direct analysis. In particular, if E is a cyclotomic extension formed by adjoining a primitive $\sqrt[k]{1}$ to the rationals, with corresponding zeta-function $\zeta_E(s)$; the explicit factorization

$$\zeta_E(s) = \prod_{\chi \bmod k} L(s, \chi) \tag{1}$$

was utilized by Siegel [1] to prove essentially that for $z = 1 + it$, the number of zeros of $\zeta_E(s)$ in the circle $|s - z| \leqq \frac{1}{2} - \varepsilon$ is bounded by $\phi(k)/(\log k)^{\delta}$ where $\delta > 0$ depends on ε. Here Siegel used the relation between the geometric and arithmetic means to reduce what appears basically as a multiplicative problem to an additive one. The orthogonality relations among the characters result in an important gain that would not otherwise be obtained, for example, by a direct application of Jensen's formula to $\zeta_E(s)$.

In recent years, important generalizations of Siegel's result have been obtained

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by Bombieri [3] and Montgomery [9]. If $N_E(\alpha, T)$, $N_\chi(\alpha, T)$ denote the number of zeros of $\zeta_E(s)$, $L(s,\chi)$ respectively in the rectangle

$$\alpha \leqq \sigma \leqq 1, \quad |t| \leqq T$$

then estimates of the form

$$N_E(\alpha, T) = \sum_{\chi \bmod k} N_\chi(\alpha, T) \ll T^{c(1-\alpha)} \qquad (T \geqq q) \tag{2}$$

have been given by Fogels [6] and generalized by Gallagher [7] to

$$\sum_{k \leqq T} \sum_{\chi \bmod k}{}^{*} N_\chi(\alpha, T) \ll T^{c(1-\alpha)} \qquad (T \geqq 1) \tag{3}$$

where * means primitive characters.

Factorizations similar to (1) occur in certain non-abelian extensions, although the L-functions can no longer be taken abelian. While it may be possible to discuss such factorizations in a more general context by considerations of intermediate fields, we shall restrict ourselves to meta-cyclic extensions. In this way, all discussion of intermediate fields is avoided, and a characterization is obtained directly through the ground field. In particular, we consider the Kummer field K_a obtained by successively adjoining (for q prime) a primitive $\sqrt[q]{1}$ and a $\sqrt[q]{a}$ for some integer $a \neq \pm 1$ or a perfect qth power. Each such field gives rise to an Artin L-function formed from a character of the representation of the meta-cyclic Galois group. Theorem 1 gives an expression for the Artin L-function directly in terms of the rational number field, and in this way, generalizations of (3) are obtained for this class of L-functions. An additional factor, however, will now depend on the degree of the character.

2. Some general notations

We let K_a be the Kummer field as described above. The nth occurrence of the letter c will denote an absolute constant c_n. For primes p and q, the symbol $\chi_{p.q}$ denotes a Dirichlet character mod p of exact order q. By $\ll$, we mean Vinogradov's symbolism for "less than a constant times".

3. The Artin L-functions

For a Galois extension K/k with non-abelian group G, a theory of L-functions has been developed by Artin [1] which is analogous to the abelian case. Here, however, representations of G into matrices over the complex numbers are considered, the characters being the traces of these matrices.

If β is a prime in K lying above some prime p in k, then the decomposition group G_β of β consists of those automorphisms $\mu \in G$ such that $\mu\beta = \beta$. The Frobenius automorphism $(\beta, K/k) = \mu$ is the unique element $\mu \in G_\beta$ characterized by the property

$$\mu a \equiv a^{Np} (\bmod \beta)$$

for all integers $a \in K$. Here Np denotes the usual norm.

For every $\mu \in G$, let $M(\mu)$ be a representation of G into matrices over the complex numbers. Let $\chi(\beta)$ be the trace of $M(\beta, K/k)$. Actually, we may write $\chi(p)$ since the value $\chi(\beta)$ is independent of $\beta \mid p$. The Artin L-function is defined by its logarithm

$$\log L(s, \chi, K/k) = \sum_{p,m} \frac{\chi(p^m)}{mNp^{ms}},$$

the sum going over primes $p \in k$ and positive rational integers m.

It was shown by Artin [2] that $L(s, \chi, K/k)$ satisfies the following properties:

(4) $L(s, \chi, K/k)$ is regular for $\sigma > 1$.

(5) $L(s, \chi_0, K/k) = \zeta_{K/k}(s)$.

(6) If $\chi = \chi_1 + \chi_2$ are characters of G, then

$$L(s, \chi, K/k) = L(s, \chi_1, K/k) \cdot L(s, \chi_2, K/k) .$$

(7) If Ω is an intermediate field between K and k so that Ω/k is normal, and if χ is a character of $\mathrm{Gal}(\Omega/k)$, then

$$L(s, \chi, K/k) = L(s, \chi, \Omega/k)$$

where χ can also be regarded as a character of G.

(8) If Ω is an intermediate field between K and k, then to each character χ of $\mathrm{Gal}(k/\Omega)$ there corresponds an induced character χ' of G such that

$$L(s, \chi', K/k) = L(s, \chi, K/\Omega).$$

It was shown by Brauer [4] that if χ is a character of G, then for rational integers n_{ij},

$$L(s, \chi, K/k) = \prod_i \prod_j L(s, \chi_{ij}, K/\Omega_i)_{ij} \tag{9}$$

where each $\mathrm{Gal}(K/\Omega_i)$ is cyclic and the χ_{ij} are abelian characters of $\mathrm{Gal}(K/\Omega_i)$. In particular, the Artin L-function $L(s, \chi, K/k)$ satisfies a functional equation induced by the functional equation of the abelian L-series in the right side of (9).

4. L-functions of Kummer fields

We consider the Kummer field $K_a = Q(\sqrt[q]{1}, \sqrt[q]{a})$ for q a prime number and $a \neq \pm 1$ or a perfect qth power. The Galois group G of K_a/Q is a metacyclic group which can be written

$$G = G_1 G_2, \qquad G_1 \cap G_2 = \langle 1 \rangle$$

where G_1 and G_2 are cyclic subgroups having orders q and $q-1$ respectively. If n is the degree of K_a/Q then $n = q(q-1)$.

The elements of G fall into q conjugacy classes, so there are only q simple characters of G, among which are included the $q-1$ linear or abelian group characters. If we denote these simple characters $\chi_1, \dots, \chi_q$, with $\chi_1, \dots, \chi_{q-1}$ linear, then it follows from the orthogonality relations that

$$\sum_{i=1}^{q} \chi_i(\mu)\bar{\chi}_i(\mu') = \begin{cases} n/l_\mu & \mu' \in \langle \mu \rangle \\ 0 & \mu' \notin \langle \mu \rangle \end{cases} \tag{10}$$

where l_μ is the order of the conjugacy class $\langle \mu \rangle$ of μ. Taking $\mu = \mu' = 1$ gives

$$\sum_{i=1}^{q} n_i^2 = n \qquad (n_i = \text{degree of } \chi_i) \tag{11}$$

so that we must have $n_q = q - 1$. Also, taking $\mu' = 1$ in (10) gives

$$\sum_{i=1}^{q-1} \chi_i(\mu) + (q-1)\chi_q(\mu) = \begin{cases} q(q-1) & \mu = 1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

and therefore, we have the factorization

$$\begin{aligned} \zeta_{K_a/Q}(s) &= L(S, \chi_0, K_a/K_a) = L\left(S, \sum_{i=1}^{q-1} \chi_i + (q-1)\chi_q, K_a/Q\right) \\ &= \left[\prod_{i=1}^{q-1} L(S, \chi_i, K_a/Q)\right] \cdot L(S, \chi_q, K_a/Q)^{(q-1)}. \end{aligned}$$

Since the characters $\chi_1, \dots, \chi_{q-1}$ may be taken as characters of G_2, it follows from (7) that with $\Omega = Q(\sqrt[q]{1})$

$$L(S, \chi_i, K/Q) = L(S, \chi_i, \Omega/Q) \qquad (1 \leqq i \leqq q-1)$$

and this is just a Dirichlet series formed with a Dirichlet character $\chi_i \bmod q$. Hence, the zeta-function of the Kummer field K_a has the following factorization:

$$\zeta_{K_a}(s) = \left[\prod_{\chi \bmod q} L(s, \chi)\right] \cdot L(s, \chi_q, K_a/Q)^{(q-1)} \tag{14}$$

where χ_q has degree $q-1$ and χ_q is induced by a character χ of $\mathrm{Gal}(K/\Omega)$. So that by (8),

$$L(s,\chi_q,K_a/Q) = L(s,\chi,K_a/\Omega).$$

In particular, the Artin L-function $L(s,\chi_q,K_a/Q)$ is regular.

The factorization (14) can be reformulated directly in terms of Dirichlet characters of the ground field Q. To establish this, it is necessary first to examine the factorization of rational primes in K. Accounts of such factorizations were originally due to Dedekind and good treatments can be found in [5, p. 91]. If p is a rational prime not dividing qa and f_1 and f_2 are minimal such that

$$p^{f_1} \equiv 1 \pmod q, \quad x^q \equiv a^{f_2} \pmod p \text{ soluble}$$

then p is unramified and factorizes in K_a as a product of $r = q(q-1)/f_1f_2$ prime ideals $\beta_1,\ldots,\beta_r$ with $N\beta_i = p^{f_1f_2}$.

Looking at the local factor L_p of $\zeta_{K_a}(s)$ corresponding to a rational prime p, we see that

$$L_p = \prod_{\beta|p}\left(1-\frac{1}{N\beta^s}\right)^{-1} = \left(1-\frac{1}{p^{f_1f_2s}}\right)^{-r}.$$

Let ξ_1,ξ_2 be primitive f_1,f_2 th roots of unity respectively. Then

$$L_p = \prod_{h_1=1}^{f_1}\prod_{h_2=1}^{f_2}\left(1-\frac{\xi_1^{h_1}\xi_2^{h_2}}{p^s}\right)^{-r}.$$

Now, as χ runs through the Dirichlet characters mod q, $\chi(p)$ takes on each value ξ^{h_1} $(h_1 = 1,\ldots,f_1)$ exactly $(q-1)/f_1$ times, and as $\chi_{p,q}^w(w = 1,\ldots,q)$ runs through the Dirichlet characters (mod p) of order q, each value $\xi_2^{h_2}(h_2 = 1,\ldots,f_2)$ is taken exactly q/f_2 times. Hence, our local factor may be taken as

$$L_p = \prod_{\chi \bmod q}\prod_{w=1}^{q}\left(1-\frac{\chi(p)\chi_{p,q}^w(a)}{p^s}\right)^{-1}.$$

It follows that $\zeta_{K_a}(s)$ has the factorization

$$\zeta_{K_a}(s) = \left[\prod_{\chi \bmod q} L(s,\chi)\right]\left[\prod_{\chi \bmod q}\prod_{w=1}^{q-1}\left(1-\frac{\chi(p)\chi_{p,q}^w(a)}{p^s}\right)^{-1}\right]. \tag{15}$$

Comparing (14) and (15) gives the following theorem.

THEOREM 1. *The Artin L-function* $L(s, \chi_q, K_a/Q)$ *may be written for* $\mathrm{Re}\, s > 1$ *as*

$$(16) \quad L(s,\chi_q,K_a/Q) = F(s)\Bigg[\prod_{\substack{p \\ p \nmid qa}} \prod_{\chi \bmod q} \prod_{w=1}^{q-1} \left(1 - \frac{\chi(p)\chi^w_{p,q}(a)}{p^s}\right)^{-1}\Bigg]^{1/(q-1)}$$

where $F(s)$ *consists of some finite product of ramified primes* $p \mid qa$.

Unfortunately, it appears as if there is no simple direct way of analytically continuing the series representation (16) to the left of the line $\mathrm{Re}(s) = 1$. Any such continuation should shed some light on the structure of a non-abelian extension in terms of the arithmetic of its ground field.

5. Application of the large sieve

Following Gallagher [7], we show that if $L(s, \chi_q, K_a/Q)$ has a zero near $z = 1 + iv$, then for suitable x, y, the sum

$$s_{x,y}(a,v) = \sum_{\substack{x \leq p \leq y \\ p \equiv 1 (\mathrm{mod}\ q)}} \sum_{w=1}^{q-1} \frac{\chi^w_{p,q}(a)}{p^z} \log p$$

is large. In this way, bounds for the number of zeros of the Artin L-functions can be determined directly from large sieve estimates for character sums. We shall prove the following theorem.

THEOREM 2. *Let* $N_a(\chi_q, \alpha, T)$ *denote the number of zeros of* $L(s,\chi_q,K_a/Q)$ *in the rectangle* $\alpha \leqq \sigma \leqq 1$, $|t| \leqq T$. *Then for positive constants* c_1, c_2, c_3, c_4, F

$$(17) \quad \sum_{a \leq A}{}' N_a(\chi_q, \alpha, T) \ll T^{c_1 n(1-\alpha)}(c_2 n \mathscr{L})^{g+F}\left[T^{2-c_3 n}A + A^{9/10 + 1/c_4 n}\right]$$

where Σ' *means* $a \neq 1$ *or a q'th power, and* $g \ll n \dfrac{\log T}{\log A}$.

Before proving (17), we first establish some lemmas.

LEMMA 1. $L(s,\chi_q,K_a/Q)$ *has* $\ll rn\mathscr{L}$, $(n = q(q-1))$ *zeros in any disc* $|s - z| \leqq r$ *provided* $(n\mathscr{L})^{-1} \leqq r \leqq 1$, $z = 1 + iv$, $|v| \leqq T$ *and* $\mathscr{L} = \log T$.

PROOF. This follows by a direct application of [10, p. 331] to the zeta function of an algebraic number field, it being noted that in this case the Artin L-function $L(s,\chi_q,K_a/Q)$ divides $\zeta_{K_a}(s)$.

LEMMA 2. *If* $L(s,\chi_q,K_a/Q)$ *has a zero in the disc* $|s - z| \leqq r$ *with* $(n\mathscr{L})^{-1} \leqq r \leqq c$, $z = 1 + iv$, $|v| \leqq T$, *then for every* $x \geqq T^{cn}$

$$\int_x^{x^B} |s_{x,y}(a,v)| \frac{dy}{y} \gg (T^{-crn}) \cdot r^2,$$

where B is a suitable constant.

PROOF. Here, we essentially follow Gallagher's argument [7]. The Artin L-function satisfies

$$\frac{L'}{L}(s,\chi_q,K_a/Q) = \sum_{\rho} \frac{1}{s-\rho} + O(n\mathscr{L}),\ |s-z| \leqq \tfrac{1}{2} \tag{19}$$

where ρ runs over zeros in $|s-z| \leqq 1$. The above is obtained most simply in some more general cases owing to the fact that the Artin L-function may divide the zeta-function of the field. An application of Cauchy's inequality to (19) gives

$$\frac{D^k}{k!}\frac{L'}{L}(s,\chi_q,K_a/Q) = (-1)^k \sum \frac{1}{(s-\rho)^{k+1}} + O(4^k n\mathscr{L}),\ |s-z| \leqq \tfrac{1}{4}.$$

The above sum contains $\ll 2^j\, n\mathscr{L}$ terms that are each $\ll (2^j\lambda)^{-(k+1)}$ for $2^j\lambda < |\rho - z| \leqq 2^{j+1}\lambda$, and their contribution is

$$\ll \sum_{j \geqq 0} (2^j\lambda)^{-k} n\mathscr{L} \ll \lambda^{-k} n\mathscr{L}.$$

Consequently, for $(n\mathscr{L})^{-1} \leqq r \leqq \lambda \leqq \frac{1}{4}$,

$$\frac{D^k}{k!}\frac{L'}{L}(z+r,\chi_q,K_a/Q) = (-1)^k {\sum}' \frac{1}{(z+r-\rho)^{k+1}} + O(\lambda^{-k} n\mathscr{L}) \tag{20}$$

where Σ' now runs over $|\rho - z| \leqq \lambda$. By Lemma 1, there are $\ll \lambda n\mathscr{L}$ such zeros ρ and $\min|z-\rho| \leqq 2r$. So by Turan's second power theorem [12]

$$\left| {\sum}' \frac{1}{(z+r-\rho)^{k+1}} \right| \geqq (Dr)^{-(k+1)}$$

for suitable constant D and for some integer $k \in [K, 2K]$ provided $K \gg \lambda n\mathscr{L}$. Hence, by choosing $\lambda = cr$, we get

$$\frac{D^k}{k!}\frac{L'}{L}(z+r,\chi_q,K_a/Q) \gg (Dr)^{-(k+1)} \quad . \tag{21}$$

Making use of the Dirichlet expansion (16), the above may be rewritten as

$$\frac{1}{q-1} \sum_{\chi \bmod q} \sum_{w=1}^{q-1} \sum_{m} \frac{\chi(m)\chi_{m,q}^{w}(a)}{m^z} \Lambda(m) P_k(r \cdot \log m)$$

$$= \sum_{m \equiv 1(q)} \sum_{w=1}^{q-1} \frac{\chi_{m,q}^{w}}{m^z} \Lambda(m) P_k(r\cdot \log m) \gg D^{-k}/r$$

where

$$P_k(u) = e^{-u}(u^k/k!)$$

and satisfies

$$P_k(u) \leqq (2D)^{-k} \text{ for } u \leqq B_1 k$$

$$P_k(u) \leqq (2D)^{-k} e^{-\frac{1}{2}u} \text{ for } u \geqq B_2 k$$

for some constants B_1 and B_2.

Let x be $\geqq T^{cn}$, with $c = B_1 E$. Put $K = B_1^{-1} r \log x$ so that $K \geqq \text{Ern}\, \mathscr{L}$, $k \in [K,\ 2K]$. It follows for $B = 2B_2/B_1$ that

$$\sum_{\substack{m \leqq x \\ m \equiv 1(q)}} \sum_{w=1}^{q-1} \frac{\chi_{m,q}(a)}{m^z} \Lambda(m) P_k(r \cdot \log m)$$

$$\ll (2D)^{-k}(q-1) \sum_{\substack{m \leqq x \\ m \equiv 1(q)}} \frac{\Lambda(m)}{m}$$

$$\ll (2D)^{-k} k/r$$

and also

$$\sum_{\substack{m \geqq x \\ m \equiv 1(q)}}{}_B \sum_{w=1}^{q-1} \frac{\chi_{m,q}(a)}{m} \Lambda(m) P_k(r \cdot \log m)$$

$$\ll (2D)^{-k}(q-1) \sum_{\substack{m \geqq x \\ m \equiv 1(q)}}{}_B \frac{\Lambda(m)}{m^{1+\frac{1}{2}r}}$$

$$\ll (2D)^{-k}/r.$$

Therefore

$$\sum_{\substack{x < m < x \\ m \equiv 1(q)}}{}_B \sum_{w=1}^{q-1} \frac{\chi_{m,q}(a)}{m} \Lambda(m) P_k(r \cdot \log m) \gg D^{-k}/r.$$

Since $P_k \ll 1$, the prime powers in (22) contribute $\ll x^{\frac{1}{2}}$ which may be ignored. Now, for $s(y) = s_{x,y}(a, r)$, we may write

$$\int_x^{xB} p_k(r \cdot \log y) ds(y) = p_k(r \cdot \log x^B) s(x^B)$$

$$- \int_x^{xB} s(y) P_k'(r \cdot \log y) r \frac{dy}{y}.$$

The first term on the right is

$$\ll (2D)^{-k}(q-1) \sum_{\substack{m \leqq xB \\ m \equiv 1(q)}} \frac{\Lambda(m)}{m} \ll (2D)^{-k} k/r,$$

and since $p_k' = p_{k-1} - p_k \ll 1$

$$\int_x^{x^B} |s(y)| \frac{dy}{y} \gg D^{-k}/r^2 .$$

LEMMA 3. *Let $y \leqq x^c$. Then the following estimate holds:*

$$(23) \qquad \sum_{a \leqq A}' |s_{x,y}(a,0)|^2 \ll A\frac{\log^2 x}{x} + \left[\left(\log\frac{y}{x}\right)^2 - \frac{1}{g}A^{9/10}(\log y)^{g+c}\right]$$

where $g \leqq 4\dfrac{\log x}{\log A} + c.$

PROOF. Let S denote the sum in the Lemma. Then since $\chi_{p_1,q}^{w_1}\,\chi_{p_2,q}^{w_2}$ can be principal only if $p_1 = p_2$, and otherwise is a primitive character $\chi \bmod p_1p_2$ of order q, it follows that

$$S \ll \frac{A\log^2 x}{x} + \sum_{\substack{x \leqq p_1, p_2 \leqq y \\ p_1 p_2 \equiv 1(q) \\ p_1 \neq p_2}} \frac{\log p_1 \log p_2}{p_1\, p_2} \sum_{\chi}'' |S(\chi)|$$

where Σ'' is over primitive characters χ mod p_1p_2 of order q, and

$$S(\chi) = \sum_{a \leqq A}' \chi(a).$$

Let T denote the double sum on the right. It now follows by Holder's inequality that $T \leqq T_1T_2$
where

$$T_1 = \left[\Sigma q \left[\frac{\log p_1 \log p_2}{p_1p_2}\right]^{2g/(2g-1)}\right]^{1-1/2g}$$

$$T_2 = \left[\Sigma \sum_{\chi}'' S(x)^{2g}\right]^{1/2g} .$$

Applying the "large sieve" estimate

$$\sum_{q \leqq Q} \sum_{\chi \bmod q}^{*} \left| \sum_{n \leqq N} a_n\chi(n) \right|^2 \ll (Q^2 + N) \sum_{n \leqq N} |a_n|^2$$

as in [8, p. 226] yields

$$T \ll \left(\log \frac{y}{x}\right)^{(2-1/g)} A^{9/10}(\log y)^{g+c}$$

which proves the lemma.

PROOF OF THEOREM. Because $N_a(\chi_q, \alpha, T) = 0$ for $|1-\alpha| \ll (n\mathscr{L})^{-1}$, it is enough to prove (17) for $|1-\alpha| \gg (n\mathscr{L})^{-1}$. It follows from Lemma 2 that if $L(s,\chi_q,K_a/Q)$ has a zero in $|s-z| \leqq |1-\alpha|$ and $x \geqq T^{cn}$ then

$$T^{cn(1-\alpha)}(n\mathscr{L})^{-3}\int_x^{x^B} |S_{x,y}(a,v)|^2 \frac{dy}{y} \gg 1.$$

There are $\ll (1-\alpha)n\mathscr{L}$ zeros in $|s-z| \leqq (1-\alpha)$ so that

$$N_a(\chi_q,\alpha,T) \ll T^{cn(1-\alpha)}(n\mathscr{L})^{-2}\int_x^{x^B}\int_{-T}^{T}|S_{x,y}(a,v)|^2\,dv\,\frac{dy}{y}$$

and therefore for some $y \in [x, x^B]$

$$\sum_{a\leqq A}{}' N_a(\chi_q,\alpha,T) \ll T^{cn(1-\alpha)}n^{-2}\mathscr{L}^{-1}\sum_{a\leqq A}{}'\int_{-T}^{T}|S_{x,y}(a,v)|^2 dv.$$

It follows by Gallagher's first theorem [7, p. 331] that

$$\sum_{a\leqq A}{}' N_a(\chi_q,\alpha,T) \ll T^{cn(1-\alpha)}n^{-2}\mathscr{L}^{-1}T^2 I.$$

where

$$I = \int_0^\infty \sum_{a\leqq A}{}' \Bigg|\sum_{\substack{y\leqq p\leqq ye^{-1/T}\\ p\equiv 1(q)}}\sum_{w=1}^{q-1}\frac{\chi^w_{p,q}(a)\log p}{p}\Bigg|^2\frac{dy}{y}. \tag{24}$$

We now apply Lemma 3 to the above, and we get

$$\int_0^\infty \sum_{a\leqq A}{}' \Bigg|\sum_{\substack{y\leqq p\leqq ye^{-1/T}\\ p\equiv 1(q)}}\sum_{w=1}^{q-1}\frac{\chi^w_{p,q}(a)\log p}{p}\Bigg|^2\frac{dy}{y} \ll \frac{A\log^3 x}{x} + (T^{-2+1/g})A^{9/10}(\log x)^{g+c} \quad . \tag{25}$$

The theorem follows from Eqs. (24) and (25).

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